Reliability of structural systems by Enhanced Monte Carlo methods

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- The concept of failure probability occurs since it is recognized that most quantities characterizing a structure and its loading conditions are subject to uncertainty. Hence random variables are used to model such quantities.
- In principle, failure probabilities can be accurately predicted by standard Monte Carlo simulation methods, but the computational burden may be prohibitive.

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- The method exploits the regularity of tail probabilities to set up an approximation procedure for the prediction of the far tail failure probabilities. It is based on estimating the failure probabilities by Monte Carlo simulation at more moderate levels.

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- That is, $\beta = \mu_M/\sigma_M$, where $\text{E}[M] = \mu_M$ and $\text{Var}[M] = \sigma_M^2$. $\mu_M = \mu_R - \mu_S$ and $\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$.

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- Hence, the failure probability $p_f(\lambda) = \mathrm{Prob}(M(\lambda) \leq 0) = \Phi(-\lambda\beta)$.

A motivating example

- Assuming that $p_f = p_f(1)$ is small, e.g. less than about 10^{-3}, it is obtained that:

$$p_f(\lambda) = \Phi(-\lambda\beta) \underset{\lambda\to 1}{\approx} \Big(\frac{1}{\lambda\beta} - \frac{1}{(\lambda\beta)^3} + \frac{3}{(\lambda\beta)^5}\Big)\,\phi(\lambda\beta)$$
$$= \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{\lambda\beta} - \frac{1}{(\lambda\beta)^3} + \frac{3}{(\lambda\beta)^5}\Big)\,\exp\Big\{-\frac{\beta^2\lambda^2}{2}\Big\},$$

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- In fact, a uniformly close approximation (with an error less than $7.5 \cdot 10^{-8}$) which is similar to the right hand side of this equation can be given for all positive values of the argument z of $\Phi(-z)$.

- For any safety margin for which a FORM or SORM approximation applies after transformation to normalized Gaussian space, it is realized that the failure probability $p_f(\lambda)$ will be given by an equation somewhat similar to the equation above.

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- A similar analysis as for the simple example cannot be easily done without making some assumptions.
- However, motivated by the simple example and the ensuing comment, we shall make the following assumption about the behaviour of the failure probability,

$$p_f(\lambda) \underset{\lambda \to 1}{\approx} q(\lambda) \exp\left\{ -a(\lambda - b)^c \right\},$$

where the function $q(\lambda)$ is slowly varying compared with the exponential function $\exp\{-a(\lambda - b)^c\}$.

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- The observation above may then be significant, as it may be easier to estimate the failure probabilities $p_f(\lambda)$ for $\lambda < 1$ accurately than the target value since they are larger and hence require less simulations.

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- The observation above may then be significant, as it may be easier to estimate the failure probabilities $p_f(\lambda)$ for $\lambda < 1$ accurately than the target value since they are larger and hence require less simulations.
- Fitting the parametric form for $p_f(\lambda)$ to the estimated values would then allow us to provide an estimate of the target value by extrapolation.

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- The flip side of such methods is the amount of computational efforts that may be involved. One of the goals of our work is to investigate to what extent the procedure discussed in the previous section can be used to ameliorate this situation.
- In this presentation we shall limit the discussion to series and parallel system reliability problems. That is, let $M_j = G_j(X_1, \dots, X_n)$, $j = 1, \dots, m$, be a set of m given safety margins expressed in terms of n basic variables. The extended class of safety margins then become $M_j(\lambda) = M_j - \mu_j(1 - \lambda)$, where $\mu_j = \mathrm{E}[M_j]$.

System reliability

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- The failure probability for the parallel system,

$$p_f(\lambda) = \text{Prob}\Big(\bigcap_{j=1}^{m} \{M_j(\lambda) \leq 0\} \Big) \, .$$

System reliability

- Any system can be written as a series system of parallel subsystems. Then,

$$p_f(\lambda) = \text{Prob}\Big(\bigcup_{j=1}^{l} \bigcap_{i \in C_j} \{M_i(\lambda) \leq 0\} \Big) \, ,$$

where each C_j is a subset of $\{1, \ldots, m\}$ for $j = 1, \ldots, l$. The C_j denote the index sets defining the parallel subsystems.

Monte Carlo based reliability estimation

- For practical estimation of the reliability, it is assumed that,

$$p_f(\lambda) \approx q \exp\left\{ -a(\lambda - b)^c \right\}, \text{ for } \lambda_0 \leq \lambda \leq 1\,,$$

  for a suitable value of λ_0, where q is now a constant parameter.

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- Hence, there is a need to identify a suitable λ_0 so that this assumption provides a good approximation of $p_f(\lambda)$ for $\lambda \in [\lambda_0, 1]$.

Monte Carlo based reliability estimation

- For a sample of size N of the vector of basic random variables $\mathbf{X} = (X_1, \ldots, X_n)$, let $N_f(\lambda)$ denote the number of samples in the failure domain of $M(\lambda)$.

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- The coefficient of variation of this estimator is

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- A fair approximation of the 95 % confidence interval for the value $p_f(\lambda)$ can be obtained as $\mathrm{CI}_{0.95}(\lambda) = \big(C^-(\lambda)\,,\ C^+(\lambda)\big)$, where

$$C^{\pm}(\lambda) = \hat{p}_f(\lambda)\big(1 \pm 1.96\, C_V(\hat{p}_f(\lambda))\big) \,.$$

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- It is now tentatively proposed to replace $q(\lambda)$ by a suitable constant value, q say, for tail values of λ.
- Hence, we will investigate the viability of the following approximation,

$$p_f(\lambda) \approx q \exp\left\{ -a(\lambda - b)^c \right\}, \ \text{ for } \ \lambda_0 \leq \lambda \leq 1 \, ,$$

for a suitable choice of λ_0.

Monte Carlo based reliability estimation

- The optimal values of the parameters q, a, b, c is found by minimizing the following mean square error function,

$$F(q, a, b, c) = \sum_{j=1}^{M} w_j \left(\log \hat{p}_f(\lambda_j) - \log q + a(\lambda_j - b)^c \right)^2 ,$$

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- w_j, $j = 1, \ldots, M$, denote weight factors that put more emphasis on the more reliable data points, alleviating the heteroscedasticity of the estimation problem at hand.
- We use $w_j = \left(\log C^+(\lambda_j) - \log C^-(\lambda_j) \right)^{-\theta}$ with $\theta = 1$ or 2, combined with a Levenberg-Marquardt least squares optimization method.

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- It is obtained that ($\overline{x} = \sum_{j=1}^{M} w_j x_j / \sum_{j=1}^{M} w_j$, $\overline{y} = \sum_{j=1}^{M} w_j y_j / \sum_{j=1}^{M} w_j$),

$$a^*(b,c) = -\frac{\sum_{j=1}^{M} w_j (x_j - \overline{x})(y_j - \overline{y})}{\sum_{j=1}^{M} w_j (x_j - \overline{x})^2} \,,$$

and

$$\log q^*(b,c) = \overline{y} + a^*(b,c)\overline{x} \,.$$

Monte Carlo based reliability estimation

- The Levenberg-Marquardt method may now be used on the function $\tilde{F}(b,c) = F(q^*(b,c), a^*(b,c), b, c)$ to find the optimal values b^* and c^*, and then the corresponding a^* and q^* can be calculated.

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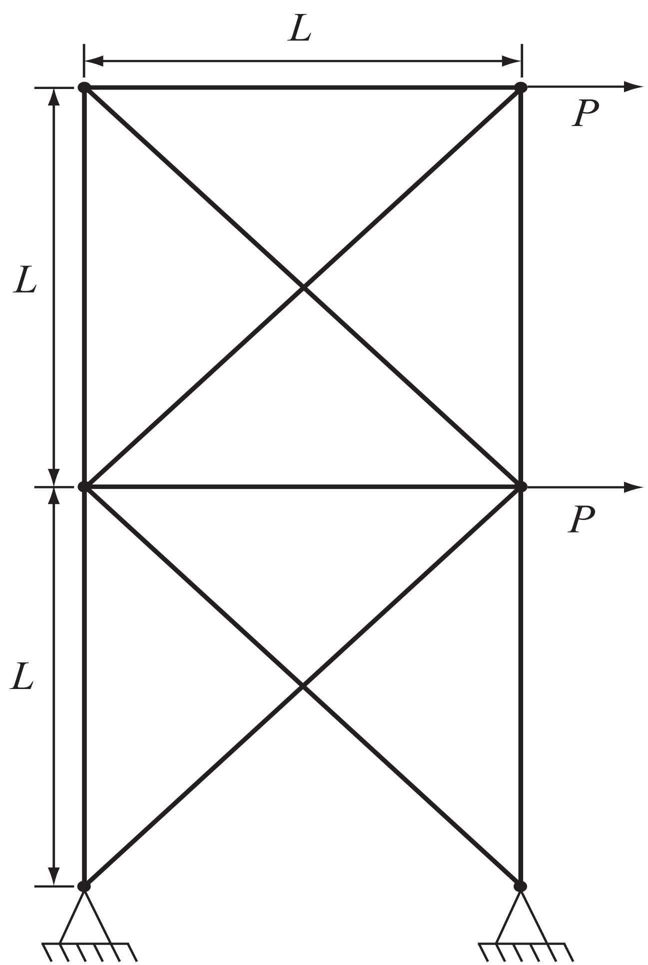
- The fitted curves to the margins of the reanchored confidence band will determine an optimized confidence interval of the predicted value. This is obtained by constrained nonlinear optimization.
- As a final point, it has been observed that the predicted value is not very sensitive to the choice of λ_0 provided it is chosen with some care.

Example 1 - Ten bar truss structure



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The safety margin for the horizontal displacement at the upper right hand corner of the truss structure with allowable displacement d_0 can be written as,

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where

$D_T = 4A_2^2(8A_1^2 + A_3^2) + 4\sqrt{2}A_1A_2A_3(3A_1 + 4A_2) + A_1A_3^2(A_1 + 6A_2)$,

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The random variable B has been introduced to account for model uncertainties. It is assumed that A_1, A_2, A_3, B, P, E are independent basic random variables.

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	Mean Value	Coef. of Var.	Prob. distr.
A_1	10^{-2} m^2	0.05	Normal
A_2	$1.5 \cdot 10^{-3}$ m^2	0.05	Normal
A_3	$6.0 \cdot 10^{-3}$ m^2	0.05	Normal
B	1.0	0.10	Normal
P	$2.5 \cdot 10^5$ N	0.10	Gumbel
E	$6.9 \cdot 10^4$ MPa	0.05	Lognormal

Table 1: Basic variables.

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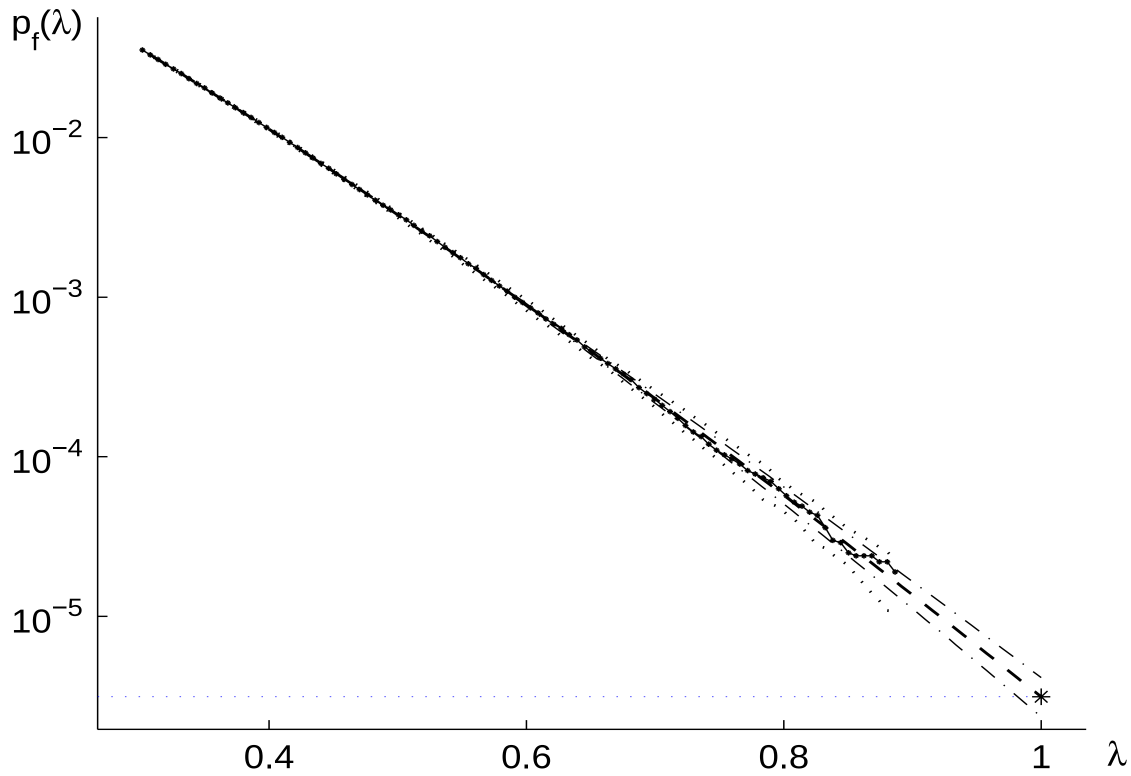
Applying the proposed procedure with a sample of size 10^6 gave the estimated value $\hat{p}_f = 3.1 \cdot 10^{-6}$ with a 95% confidence interval $(2.4 \cdot 10^{-6}, 4.1 \cdot 10^{-6})$.
The CPU time for these results were about two minutes on a standard laptop.

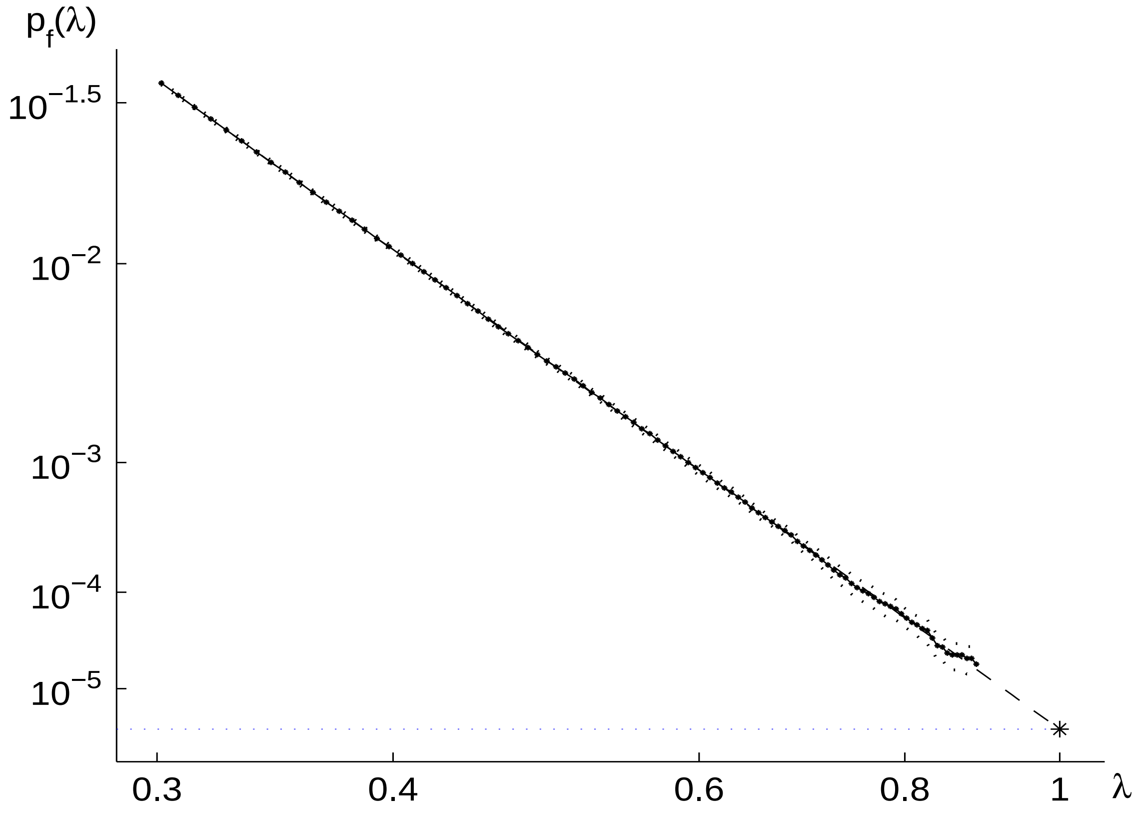
Example 1 - Ten bar truss structure

Plot of $\log \hat{p}_f(\lambda_j)$: Monte Carlo (•); fitted optimal curve (– –); reanchored empirical confidence band ($\cdot\cdot\cdot$); fitted confidence band (– ·).



Example 1 - Ten bar truss structure

Optimal double log plot of $\hat{p}_f(\lambda_j)/q$: Monte Carlo (•); fitted optimal curve (– –); empirical confidence band (- -).



Example 2 - System with equi-correlated safety margins

In this example we shall use the following simple safety margins,

$$M_j = R_j - S\,,\;\; j = 1, \ldots m\,,$$

where $R_j \sim N(\mu_R, \sigma_R^2)$, $j = 1, \ldots, m$, and $S \sim N(\mu_S, \sigma_S^2)$, and all random variables are assumed to be independent.

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The correlation coefficient ρ_{ij} between M_i and M_j is given as follows, which is due to the shared loading for all safety margins,

$$\rho_{ij} = \rho = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_R^2}\,, \;\; i \neq j\,, \;\; i, j = 1, \ldots m\,.$$

Example 2 - System with equi-correlated safety margins

In this example we shall use the following simple safety margins,

$$M_j = R_j - S\,,\ \ j = 1, \dots m\,,$$

where $R_j \sim N(\mu_R, \sigma_R^2)$, $j = 1, \dots, m$, and $S \sim N(\mu_S, \sigma_S^2)$, and all random variables are assumed to be independent.
The correlation coefficient ρ_{ij} between M_i and M_j is given as follows, which is due to the shared loading for all safety margins,

$$\rho_{ij} = \rho = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_R^2}\,,\ \ i \neq j\,,\ \ i, j = 1, \dots m\,.$$

Hence, the safety margins are equi-correlated. It is clear that the extended class of safety margins $M_j(\lambda) = M_j - \mu_j(1 - \lambda)$, $j = 1, \dots, m$ remain equi-correlated with the same correlation coefficient ρ.

Example 2 - System with equi-correlated safety margins

We now consider the series system with failure determined by $\bigcup_{j=1}^{m}\{M_j(\lambda) \leq 0\}$. The failure probability can be expressed as,

$$p_f(\lambda) = 1 - \int_{-\infty}^{\infty} \phi(t) \left[\Phi\Big(\frac{\beta\lambda - \sqrt{\rho}\, t}{\sqrt{1-\rho}} \Big) \right]^m dt \,,$$

where β denotes the common safety index of the M_j given by $\beta = (\mu_R - \mu_S)/\sqrt{\sigma_R^2 + \sigma_S^2}$.

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For the numerical example, $m = 10$, $\sigma_R^2 = \sigma_S^2 = 0.5$, $\mu_S = 5.0$, $\mu_R = \beta + 5.0$, and $\beta = 4.0, 4.5, 5.0$.

Example 2 - System with equi-correlated safety margins

β	Exact value	Estimated value	95% conf. int.	Sample size
4.0	$3.0 \cdot 10^{-4}$	$3.1 \cdot 10^{-4}$	$(2.5,\ 3.7) \cdot 10^{-4}$	10^5
4.5	$3.3 \cdot 10^{-5}$	$2.8 \cdot 10^{-5}$	$(2.4,\ 3.4) \cdot 10^{-5}$	10^5
5.0	$2.8 \cdot 10^{-6}$	$2.6 \cdot 10^{-6}$	$(2.2,\ 3.0) \cdot 10^{-6}$	$5 \cdot 10^5$

Table 2: Failure probability for series system.

Example 2 - System with equi-correlated safety margins

We now consider the parallel system with failure given by $\bigcap_{j=1}^{m}\{M_j(\lambda) \leq 0\}$. The failure probability is given as,

$$p_f(\lambda) = \int_{-\infty}^{\infty} \phi(t) \left[\Phi\Big(\frac{-\beta\lambda - \sqrt{\rho}\, t}{\sqrt{1-\rho}} \Big) \right]^m dt \, .$$

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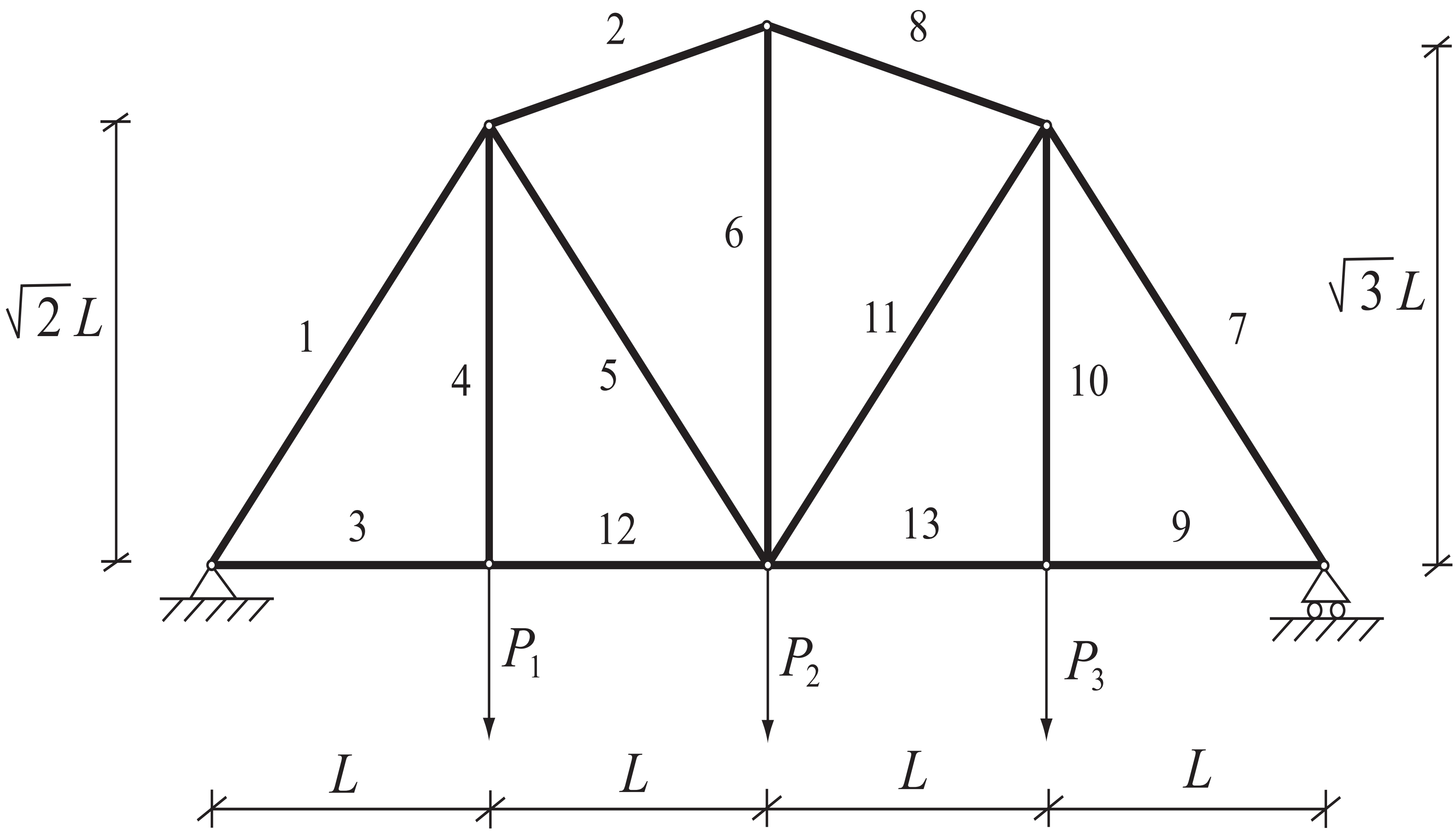
The results for $\beta = 2.0, 2.5, 3.0$ were calculated. The smaller β-values were chosen to get a reasonable probability range for system failure.

Example 2 - System with equi-correlated safety margins

β	Exact value	Estimated value	95% conf. int.	Sample size
2.0	$5.7 \cdot 10^{-5}$	$4.9 \cdot 10^{-5}$	$(2.7,\ 7.7) \cdot 10^{-5}$	10^5
2.5	$3.4 \cdot 10^{-6}$	$3.3 \cdot 10^{-6}$	$(2.0,\ 5.2) \cdot 10^{-5}$	$5 \cdot 10^5$
3.0	$1.4 \cdot 10^{-7}$	$1.3 \cdot 10^{-7}$	$(0.6,\ 2.4) \cdot 10^{-7}$	10^6

Table 3: Failure probability for parallel system.

Example 3 - Truss bridge structure



Example 3 - Truss bridge structure

	Mean Value	Coef. of Var.	Prob. distr.
$\sigma_{yj}\ ,\ j = 1, \ldots, 13$	275.8 MPa	0.15	Normal
$P_j\ ,\ j = 1, 2, 3$	89 kN	0.15	Normal

Table 4: The 16 basic variables for Example 4.

Example 3 - Truss bridge structure

$$
\begin{aligned}
M_1 &= R_1 - 0.9186P_1 - 0.6124P_2 - 0.3062P_3 \\
M_2 &= R_2 - 0.3029P_1 - 0.6058P_2 - 0.3029P_3 \\
M_3 &= R_3 - 0.5303P_1 - 0.3535P_2 - 0.1768P_3 \\
M_4 &= R_4 - P_1 \\
M_5 &= R_5 + 0.4186P_1 - 0.3876P_2 - 0.1938P_3 \\
M_6 &= R_6 - 0.1835P_1 - 0.3670P_2 - 0.1835P_3 \\
M_7 &= R_7 - 0.3062P_1 - 0.6124P_2 - 0.9186P_3 \\
M_8 &= R_8 - 0.3029P_1 - 0.6058P_2 - 0.3029P_3 \\
M_9 &= R_9 - 0.1768P_1 - 0.3535P_2 - 0.5303P_3 \\
M_{10} &= R_{10} - P_1 \\
M_{11} &= R_{11} - 0.1938P_1 - 0.3876P_2 + 0.4186P_3 \\
M_{12} &= R_{12} - 0.5303P_1 - 0.3536P_2 - 0.1768P_3 \\
M_{13} &= R_{13} - 0.1768P_1 - 0.3536P_2 - 0.5303P_3
\end{aligned}
$$

Example 3 - Truss bridge structure

The failure probability of the system was first calculated by crude Monte Carlo simulation with $2.24 \cdot 10^9$ samples. This gave the result $p_f^{MC} = 2.7 \cdot 10^{-5}$, which is accurate within about $\pm 0.5\%$ with 95% confidence. The required computation time for this was about 17 hours on a laptop computer.

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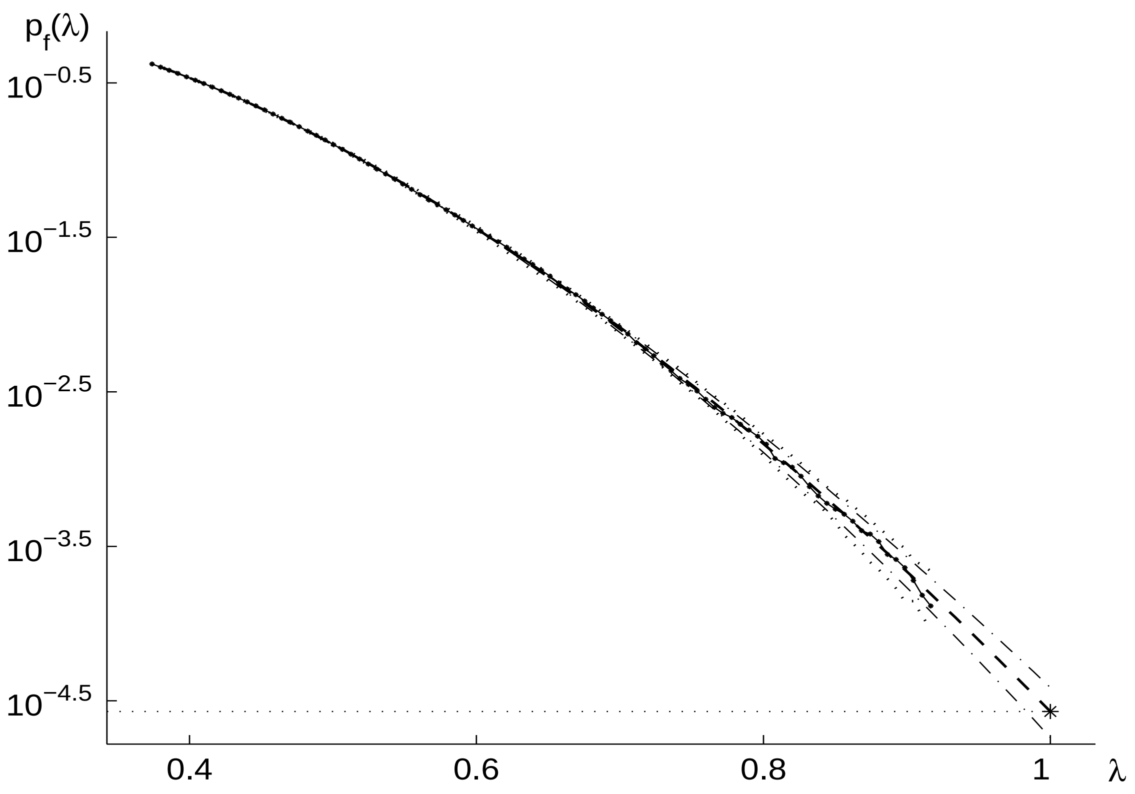
Applying the proposed procedure with a sample of size 10^5 gave the estimated value $\hat{p}_f = 2.7 \cdot 10^{-5}$ with a 95% confidence interval $(1.8 \cdot 10^{-5},\ 3.9 \cdot 10^{-5})$. The CPU time for these results were about 10 seconds on a standard laptop.

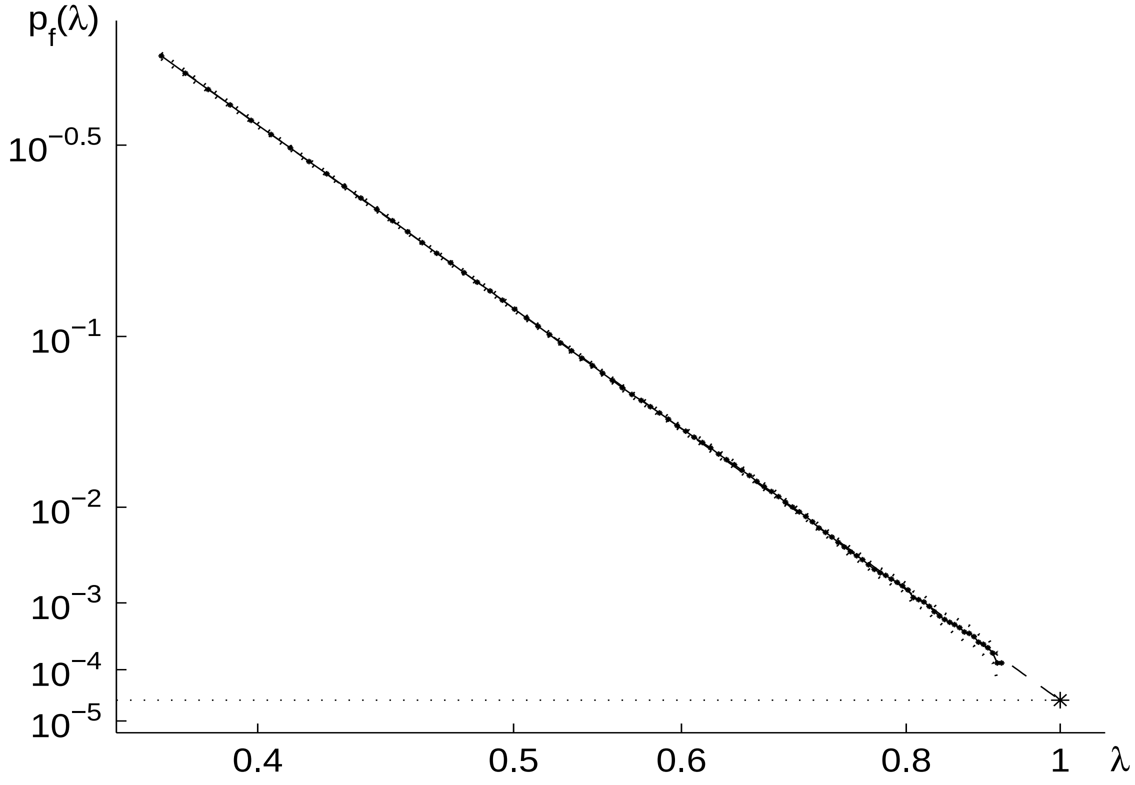
Example 3 - Truss bridge structure

Plot of $\log \hat{p}_f(\lambda_j)$: Monte Carlo ($\bullet$); fitted optimal curve (– –); reanchored empirical confidence band ($\cdot\cdot\cdot$); fitted confidence band (– $\cdot$).

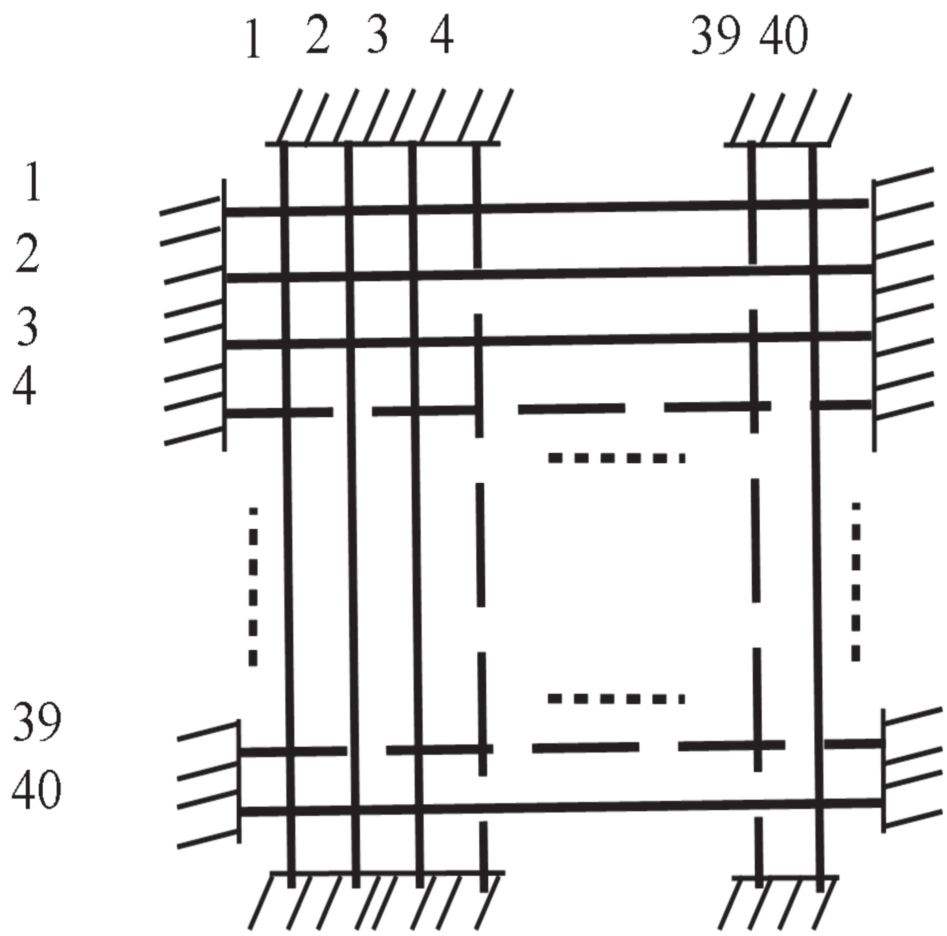


Example 3 - Truss bridge structure

Optimal double log plot of $\hat{p}_f(\lambda_j)/q$: Monte Carlo (•); fitted optimal curve (– –); empirical confidence band (- -).



Example 4 - Grillage structure



Example 4 - Grillage structure

- The grillage structure is modelled as a series system with 6540 limit state functions and a total of 4880 random variables. Each beam element has a rectangular cross section of dimension 0.05 m times 0.12 m and a length of 0.5 m.

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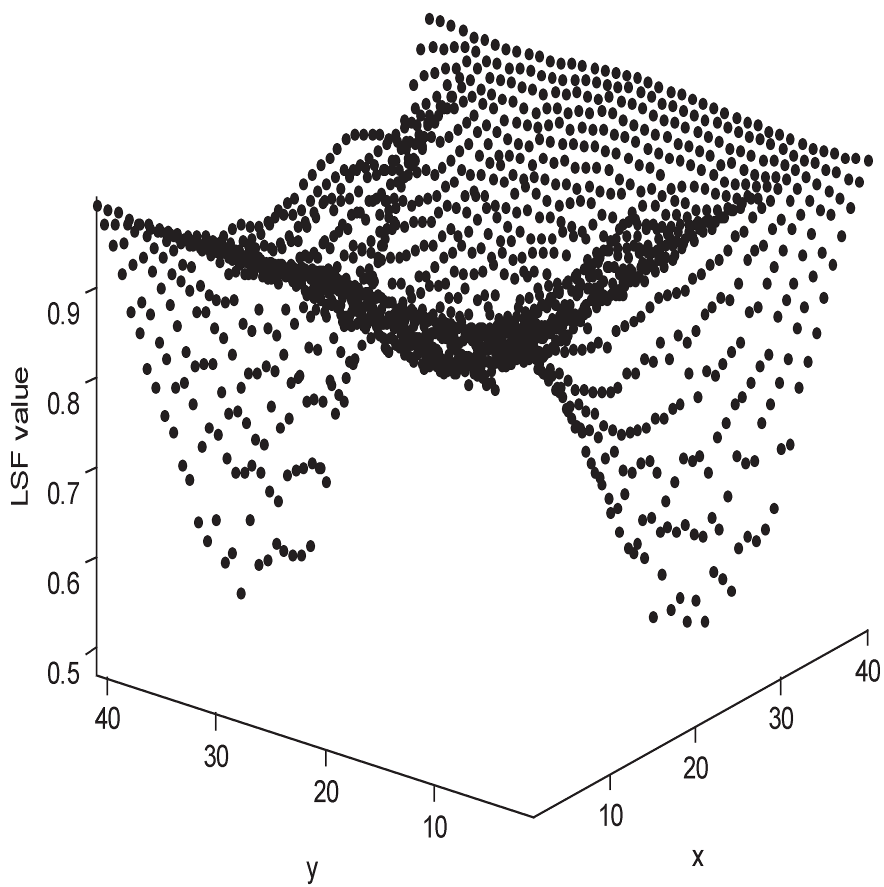
- All vertical load components are normally distributed with mean 420N and standard deviation 126N. They are pairwise equicorrelated. The yield stress is modelled as lognormally distributed random variables, mean = 380 MPa, standard deviation = 19 MPa
- The limit state function for each node of the grillage has been formulated as

$$g(\mathbf{X}) = 1 - \left\{ \left|\frac{\sigma_B}{\sigma_{B,cr}}\right| + \left|\frac{\tau_V}{\tau_{V,cr}}\right| + \left|\frac{\tau_T}{\tau_{T,cr}}\right| \right\}$$

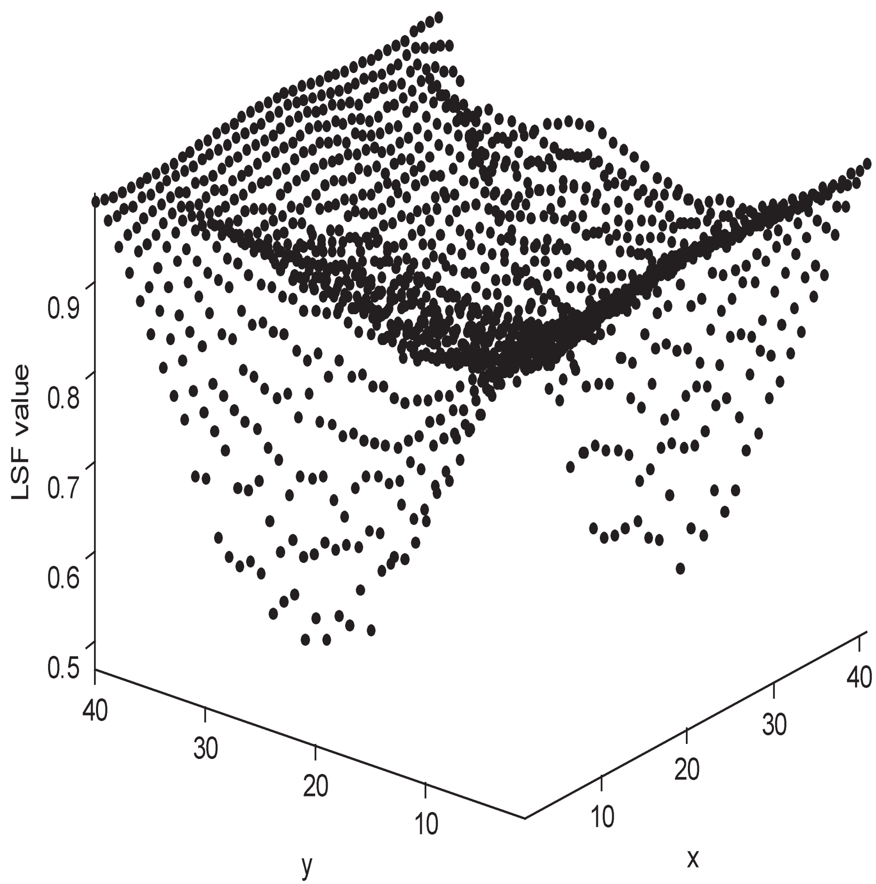
Example 4 - Grillage structure

Limit state function values for Node 2 of transverse beam elements.



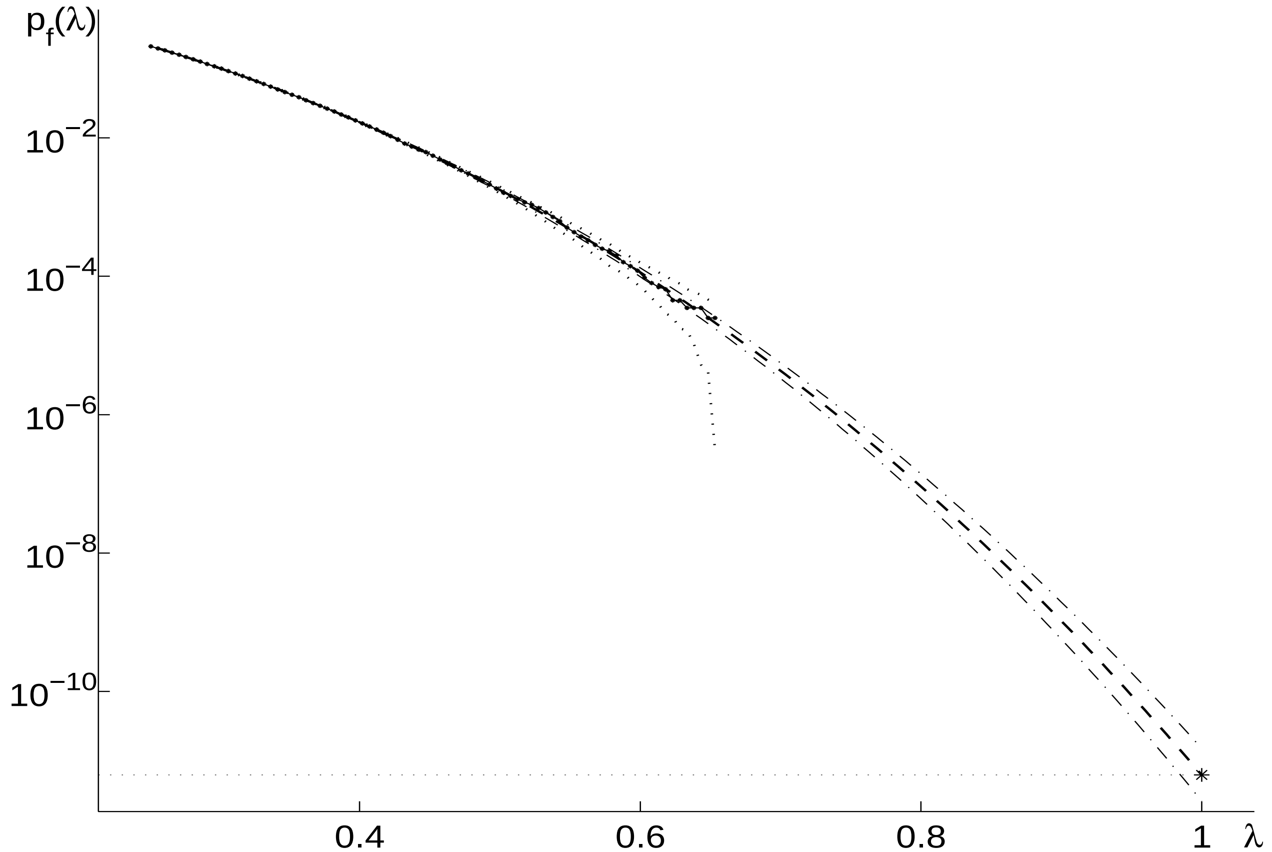
Example 4 - Grillage structure

Limit state function values for Node 2 of longitudinal beam elements.



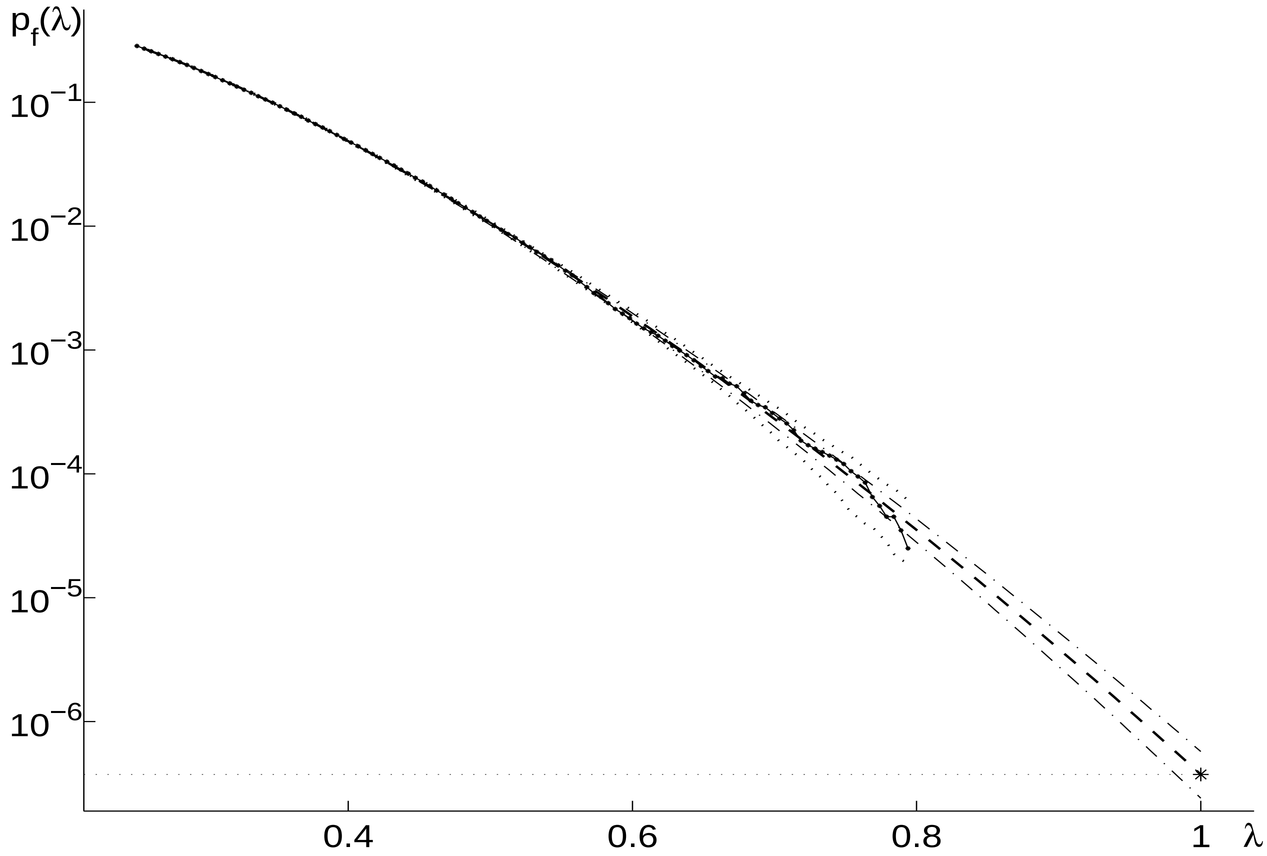
Example 4 - Grillage structure

Plot of the probability of failure p_f versus λ for load correlation length 1.5m: Monte Carlo (●); fitted optimal curve (– –); reanchored empirical confidence band ($\cdot\cdot\cdot$); fitted confidence band (– $\cdot$).



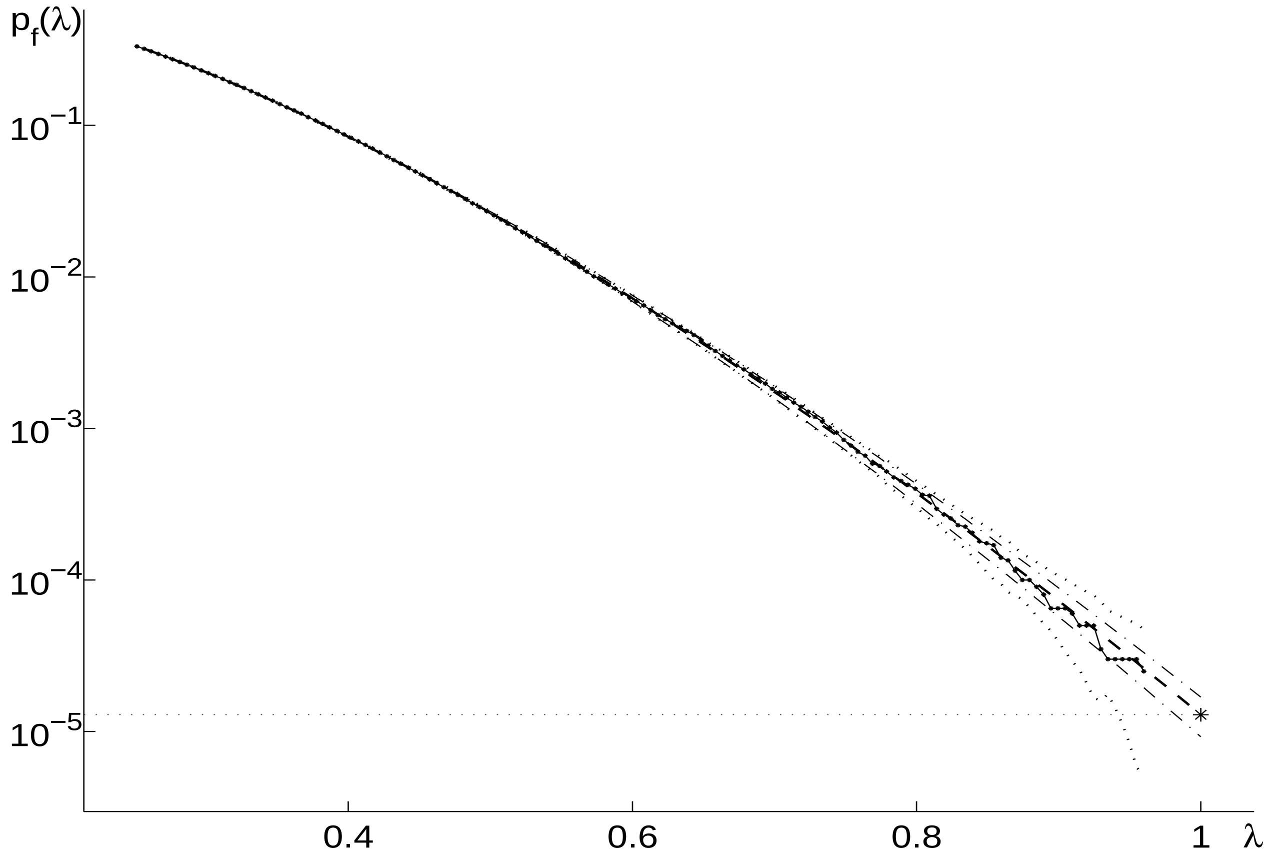
Example 4 - Grillage structure

Plot of the probability of failure p_f versus λ for load correlation length 2.0m: Monte Carlo ($\bullet$); fitted optimal curve (– –); reanchored empirical confidence band ($\cdot\cdot\cdot$); fitted confidence band (– $\cdot$).



Example 4 - Grillage structure

Plot of the probability of failure p_f versus λ for load correlation length 2.5m: Monte Carlo ($\bullet$); fitted optimal curve (– –); reanchored empirical confidence band ($\cdot\cdot\cdot$); fitted confidence band (– $\cdot$).



Conclusions

- From the examples studied, one can (tentatively) conclude that the proposed Monte Carlo based method for system reliability calculations appears to be accurate (enough) and robust, while it is simple and practical to use.

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- The CPU time is in all examples tractable, and it is reduced by a factor of ≥ 100, compared to crude Monte Carlo simulations down to the target failure probability levels.